VI. *On the Forty-eight Coordinates of a Cubic Curve in Space.*

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IN a note published in the Report of the British Association for 1878 (Dublin), and in a fuller paper in the Transactions of the London Mathematical Society, 1879 (vol. x., No. 152), I have given the forms of the eighteen, or the twenty-one (as there explained), coordinates of a conic in space, corresponding, so far as correspondence subsists, with the six coordinates of a straight line in space; and in the same papers I have established the identical relations between these coordinates, whereby the number of independent quantities is reduced to eight, as it should be. In both cases, viz.: the straight line and the conic, the coordinates are to be obtained by eliminating the variables in turn from the two equations representing the line or the conic, and are in fact the coefficients of the equations resulting from the eliminations.

In the present paper I have followed the same procedure for the case of a cubic curve in space. Such a curve may, as is well known, be regarded as the intersection of two quadric surfaces having a generating line in common; and the result of the elimination of any one of the variables from two quadric equations satisfying this condition is of the third degree. The number of coefficients so arising is $4\times10=40$; but I have found that these forty quantities may very conveniently be replaced by forty-eight others, which are henceforward considered as the coordinates of the cubic curve in space. The relation between the forty and the forty-eight coordinates is as follows: on examining the equations resulting from the eliminations of the variables, it turns out that they can be rationally transformed into expressions such as $UP'-U'P=0$, where U and U′ are quadrics, and P and P′ linear functions of the variables remaining after the eliminations. The forty-eight coordinates then consist of the twenty-four coefficients of the four functions of the form U (say the U coordinates) together with the twenty-four coefficients of the functions of the form U′ (say the U′ coordinates), arising from the four eliminations respectively; viz.: $4\times6+4\times6=48$. And it will be found that the coefficients of the forms P, P′, are already comprised among those of U, U′; so that they do not add to the previous total of forty-eight.

The number of identical relations established in the present paper is 34. But it will be observed that the equations, $UQ'-U'P=0$, are lineo-linear in the U coordinates and in the U′ coordinates; and as we are concerned with the ratios only of the coefficients and not with their absolute values, we are in fact concerned only with the ratios of the U coordinates *inter se*, and of the U′ coordinates *inter se*, and not with their absolute values. Hence the number of independent coordinates will be reduced to $48-34-2=12$, as it should be.

§ 1. *Formation of the equations.*

With a view to the problem in question, it is first required to form the equations of two quadrics having a generating line in common. For the present purpose the following appears the simplest way of effecting this; let

$$\begin{aligned} u&=\alpha x+\beta y+\gamma z+\delta t, & u'&=\alpha' x+\beta' y+\gamma' z+\delta' t, \\ v&=\alpha_1 x+\beta_1 y+\gamma_1 z+\delta_1 t, & v'&=\alpha_1' x+\beta_1' y+\gamma_1' z+\delta_1' t, \\ w&=\alpha_2 x+\beta_2 y+\gamma_2 z+\delta_2 t, & w'&=\alpha_2' x+\beta_2' y+\gamma_2' z+\delta_2' t, \end{aligned} \quad (1)$$

From these we may form the three equations

$$\begin{aligned} vw'-v'w&=(a,\ b,\ \ldots)(x,\ y,\ z,\ t)^2=0, \\ wu'-w'u&=(a_1,\ b_1,\ \ldots)(x,\ y,\ z,\ t)^2=0, \\ uv'-u'v&=(a_2,\ b_2,\ \ldots)(x,\ y,\ z,\ t)^2=0, \end{aligned} \quad (2)$$

of which two only, of course, are independent. Any two of them may be taken as representing the two quadrics in question. Thus, if we take the last two, the equations of the common generating line will be $u=0$, $u'=0$.

The next step is to eliminate the four variables in turn from the two quadrics; for which purpose it will be convenient to express the system (2) in the following form:

$$u:u'=v:v'=w:w'; \quad (3)$$

or, introducing an indeterminate quantity θ, we may use, instead of equations (3), the following, viz.:

$$u+\theta u'=0, \quad v+\theta v'=0, \quad w+\theta w'=0. \quad (4)$$

In order to eliminate one of the variables, say t, let us write,

$$\begin{aligned} u&=u_t+\delta t, & u'&=u_t'+\delta' t, \\ v&=v_t+\delta_1 t, & v'&=v_t'+\delta_1' t, \\ w&=w_t+\delta_2 t, & w'&=w_t'+\delta_2' t. \end{aligned} \quad (5)$$

The system (4) will then take the form

$$\begin{aligned} u_t+\delta t+\theta u_t'+\theta t\delta' &=0, \\ v_t+\delta_1 t+\theta v_t'+\theta t\delta_1' &=0, \\ w_t+\delta_2 t+\theta w_t'+\theta t\delta_2' &=0; \end{aligned}$$

or, as it may be written,

$$\begin{array}{llllll} 0=u+\delta t+\theta u'+\theta t\delta' & (u,\ u', \text{ suffix } t; & \delta,\ \delta', \text{ suffix } 0) & . & . & . \quad . \quad (6) \\ 0=v+\delta t+\theta v'+\theta t\delta' & (v,\ v',\ \ ,, \quad t; & \delta,\ \delta',\ \ ,, \quad 1), & & & \\ 0=w+\delta t+\theta w'+\theta t\delta' & (w,\ w',\ \ ,, \quad t; & \delta,\ \delta',\ \ ,, \quad 2), & & & \end{array}$$

and if we multiply these equations throughout by t, we may write the two systems (6), and $t(6)$ thus:

$$\begin{array}{ll} u+\theta u'+t\delta+\theta t\delta' \quad . \quad . \quad . \quad =0, & (u,\ u', \text{ suffix } t;\ \delta,\ \delta', \text{ suffix } 0) \quad . \quad . \quad (7) \\ v+\theta v'+t\delta+\theta t\delta' \quad . \quad . \quad . \quad =0, & (v,\ v',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 1) \\ w+\theta w'+t\delta+\theta t\delta' \quad . \quad . \quad . \quad =0, & (w,\ w',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 2) \\ . \quad . \quad . \quad tu+\theta tu'+t^2\delta+\theta t^2\delta'=0, & (u,\ u',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 0) \\ . \quad . \quad . \quad tv+\theta tv'+t^2\delta+\theta t^2\delta'=0, & (v,\ v',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 1) \\ . \quad . \quad . \quad tw+\theta tw'+t^2\delta+\theta t^2\delta'=0, & (w,\ w',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 2) \end{array}$$

whence we may at once eliminate 1, θ, t, θt, t^2, θt^2, and obtain the following result:

$$\begin{array}{ll} u,\ u',\ \delta,\ \delta',\ .\ . \ =0, & (u,\ u', \text{ suffix } t;\ \delta,\ \delta', \text{ suffix } 0) \quad . \quad . \quad . \quad . \quad (8) \\ v,\ v',\ \delta,\ \delta',\ .\ . & (v,\ v',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 1) \\ w,\ w',\ \delta,\ \delta',\ .\ . & (w,\ w',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 2) \\ .\ .\ u,\ u',\ \delta,\ \delta' & (u,\ u',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 0) \\ .\ .\ v,\ v',\ \delta,\ \delta' & (v,\ v',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 1) \\ .\ .\ w,\ w',\ \delta,\ \delta' & (w,\ w',\ \ ,, \quad t;\ \delta,\ \delta',\ \ ,, \quad 2) \end{array}$$

The corresponding results, when x, y, z are respectively eliminated, are obvious; viz.: writing down the upper lines only, they may be represented thus:

$$\begin{array}{ll} |\ u,\ u',\ \alpha,\ \alpha',\ .\ .\ | =0, & (u,\ u', \text{ suffix } x) \quad . \quad . \quad . \quad . \quad . \quad (8') \\ |\ u,\ u',\ \beta,\ \beta',\ .\ .\ | =0, & (u,\ u',\ \ ,, \quad y) \\ |\ u,\ u',\ \gamma,\ \gamma',\ .\ .\ | =0, & (u,\ u',\ \ ,, \quad z) \end{array}$$

From these equations it is not difficult to write down the coefficients of the powers and products of the variables in each case. Thus in the case of (8), or t eliminated, we shall have, for $[x^3]$, . . . $3[x^2y]$, . . . , *i.e.*, the coefficients of x^3, . . . $3x^2y$, . . . respectively,

$$
\begin{array}{lll}
[x^3]=\alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & (\text{suffix } 0) & \qquad (9)\\
\quad\ \alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & (\ \ ,,\ \ 1) & \\
\quad\ \alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & (\ \ ,,\ \ 2) & \\
\quad\ .\ .\ \alpha,\ \alpha',\ \delta,\ \delta', & (\ \ ,,\ \ 0) & \\
\quad\ .\ .\ \alpha,\ \alpha',\ \delta,\ \delta', & (\ \ ,,\ \ 1) & \\
\quad\ .\ .\ \alpha,\ \alpha',\ \delta,\ \delta', & (\ \ ,,\ \ 2) &
\end{array}
$$

$$
\begin{array}{llll}
3[x^2y]=\alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & +\alpha,\ \beta',\ \delta,\ \delta',\ .\ . & +\beta,\ \alpha',\ \delta,\ \delta',\ .\ . & (\text{suffix } 0)\\
\quad\ \alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & \ \ \alpha,\ \beta',\ \delta,\ \delta',\ .\ . & \ \ \beta,\ \alpha',\ \delta,\ \delta',\ .\ . & (\ \ ,,\ \ 1)\\
\quad\ \alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & \ \ \alpha,\ \beta',\ \delta,\ \delta',\ .\ . & \ \ \beta,\ \alpha',\ \delta,\ \delta',\ .\ . & (\ \ ,,\ \ 2)\\
\quad\ .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & (\ \ ,,\ \ 0)\\
\quad\ .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & (\ \ ,,\ \ 1)\\
\quad\ .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & (\ \ ,,\ \ 2)
\end{array}
$$

But there is another and very useful form which may be given to the equations (8) and (8′). In fact, writing down only the first lines of the determinants, and putting

$$
\begin{array}{lll}
|\,u,\ u',\ \alpha\,|=\mathrm{U}, & |\,u,\ u',\ \alpha'\,|=\mathrm{U}', & (u,\ u',\ \text{suffix } x) \qquad (10)\\
|\,u,\ u',\ \beta\,|=\mathrm{V}, & |\,u,\ u',\ \beta'\,|=\mathrm{V}', & (u,\ u',\ \ ,,\ \ y)\\
|\,u,\ u',\ \gamma\,|=\mathrm{W}, & |\,u,\ u',\ \gamma'\,|=\mathrm{W}', & (u,\ u',\ \ ,,\ \ z)\\
|\,u,\ u',\ \delta\,|=\mathrm{T}, & |\,u,\ u',\ \delta'\,|=\mathrm{T}', & (u,\ u',\ \ ,,\ \ t)\\
|\,u,\ \alpha,\ \alpha'\,|=\mathrm{P}, & |\,u',\ \alpha,\ \alpha'\,|=\mathrm{P}', & (u,\ u',\ \ ,,\ \ x)\\
|\,u,\ \beta,\ \beta'\,|=\mathrm{Q}, & |\,u',\ \beta,\ \beta'\,|=\mathrm{Q}', & (u,\ u',\ \ ,,\ \ y)\\
|\,u,\ \gamma,\ \gamma'\,|=\mathrm{R}, & |\,u',\ \gamma,\ \gamma'\,|=\mathrm{R}', & (u,\ u',\ \ ,,\ \ z)\\
|\,u,\ \delta,\ \delta'\,|=\mathrm{S}, & |\,u',\ \delta,\ \delta'\,|=\mathrm{S}', & (u,\ u',\ \ ,,\ \ t)
\end{array}
$$

We may write the results of eliminating x, y, z, t respectively in the following forms:

$$
\begin{array}{l}
\mathrm{UP}'-\mathrm{U}'\mathrm{P}=0 \qquad (11)\\
\mathrm{VQ}'-\mathrm{V}'\mathrm{Q}=0,\\
\mathrm{WR}'-\mathrm{W}'\mathrm{R}=0,\\
\mathrm{TS}'-\mathrm{T}'\mathrm{S}=0;
\end{array}
$$

and these may be regarded either as the equations of the projections of the curve on the four coordinate planes respectively, or as equations of conical surfaces passing through the curve, and having their axes parallel to the coordinate axes respectively.

It might at first sight appear that, in each of the curves (11), we should in general have nine points determined as follows: viz. (taking the last equation)

by the intersection	of	S	and	S′, one point,
,,	,,	T	,,	S, two points,
,,	,,	T′	,,	S′, two points,
,,	,,	T	,,	T′, four points.

nine in all. This however is not strictly the case, for

$$\begin{array}{rllr} \text{S}', \text{S}, \text{T}', \text{T} = u, u', \delta, \delta', & (u, u', \text{suffix } t; \ \delta, \delta', \text{suffix } 0) & & (12) \\ v, v', \delta, \delta', & (v, v', \quad ,, \quad t; \ \delta, \delta', \quad ,, \quad 1) & & \\ w, w', \delta, \delta', & (w, w', \quad ,, \quad t; \ \delta, \delta', \quad ,, \quad 2) & & \end{array}$$

and since the four determinants which can be formed from the above matrix are equivalent to only two independent determinants, it follows that if any two of the equations S=0, S′=0, T=0, T′=0, are satisfied, all will be satisfied. In other words, the four curves S, S′, T, T′, have a common point of intersection.

Instead, however, of taking the coefficients of the equations (8) and (8′), *i.e.*, the quantities of which (9) are specimens, as the forty coordinates of the curve, it will be convenient to take the coefficients of the functions (10) as the coordinates. These will be found to be forty-eight in number, and to be comprised in the following table, as stated in the introductory remarks. For the sake of brevity, only the upper lines of the determinants are written down:

$$\begin{array}{ll} \beta, \beta', \alpha = \text{B}, \ (\text{suffix } x), & \beta, \delta', \alpha + \delta, \beta', \alpha = 2\text{M} \ (\text{suffix } x), \quad . \quad . \quad (13) \\ \gamma, \gamma', \alpha = \text{C}, \ (\ ,, \ x), & \gamma, \delta', \alpha + \delta, \gamma', \alpha = 2\text{N} \ (\ ,, \ x), \\ \delta, \delta', \alpha = \text{D}, \ (\ ,, \ x), & \beta, \gamma', \alpha + \gamma, \beta', \alpha = 2\text{F} \ (\ ,, \ x), \\ & \\ \alpha, \alpha', \beta = \text{A}, \ (\ ,, \ y), & \alpha, \delta', \beta + \delta, \alpha', \beta = 2\text{L} \ (\ ,, \ y), \\ \gamma, \gamma', \beta = \text{C}, \ (\ ,, \ y), & \gamma, \delta', \beta + \delta, \gamma', \beta = 2\text{N} \ (\ ,, \ y), \\ \delta, \delta', \beta = \text{D}, \ (\ ,, \ y), & \gamma, \alpha', \beta + \alpha, \gamma', \beta = 2\text{G} \ (\ ,, \ y), \\ & \\ \alpha, \alpha', \gamma = \text{A}, \ (\ ,, \ z), & \alpha, \delta', \gamma + \delta, \alpha', \gamma = 2\text{L} \ (\ ,, \ z), \\ \beta, \beta', \gamma = \text{B}, \ (\ ,, \ z), & \beta, \delta', \gamma + \delta, \beta', \gamma = 2\text{M} \ (\ ,, \ z), \\ \delta, \delta', \gamma = \text{D}, \ (\ ,, \ z), & \alpha, \beta', \gamma + \beta, \alpha', \gamma = 2\text{H} \ (\ ,, \ z), \\ & \\ \alpha, \alpha', \delta = \text{A}, \ (\ ,, \ t), & \beta, \gamma', \delta + \gamma, \beta', \delta = 2\text{F} \ (\ ,, \ t), \\ \beta, \beta', \delta = \text{B}, \ (\ ,, \ t), & \gamma, \alpha', \delta + \alpha, \gamma', \delta = 2\text{G} \ (\ ,, \ t), \\ \gamma, \gamma', \delta = \text{C}, \ (\ ,, \ t), & \alpha, \beta', \delta + \beta, \alpha', \delta = 2\text{H} \ (\ ,, \ t), \end{array}$$

with a similar set of expressions of which the following are specimens :

$$\beta,\ \beta',\ \alpha'=B'_x,\quad \beta,\ \delta',\ \alpha'+\delta,\ \beta',\ \alpha'=2M'_x \quad \ldots \quad (14)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Substituting these values in (10), we shall find the following to be the forms of the functions therein contained, viz. :

$$\begin{array}{ll}
U=(B, C, D, N, M, F)(y, z, t)^2, & U'=(B', \ldots)(y, z, t)^2;\ (B, \ldots \text{ suffix } x) \quad (15)\\
V=(C, A, D, L, N, G)(z, x, t)^2, & V'=(C', \ldots)(z, x, t)^2;\ (C, \ldots \quad ,, \quad y)\\
W=(A, B, D, M, L, H)(x, y, t)^2, & W'=(A', \ldots)(x, y, t)^2;\ (A, \ldots \quad ,, \quad z)\\
T=(A, B, C, F, G, H)(x, y, z)^2, & T'=(A', \ldots)(x, y, z)^2;\ (A, \ldots \quad ,, \quad t)\\
P=A_y y+A_z z+A_t t, & P'=A_y' y+\ldots)\\
Q=B_x x+B_z z+B_t t, & Q'=B_x' x+\ldots)\\
R=C_x x+C_y y+C_t t, & R'=C_x' x+\ldots)\\
S=D_x x+D_y y+D_z z, & S'=D_x' x+\ldots)
\end{array}$$

And finally we may, by means of the equations (11), replace forms having the coefficients (9), by forms having coefficients into which the quantities A, B, ... enter. Thus, writing

$$B_xA_y'-B_x'A_y=(B, A, x, y),\quad C_xA_z'-C_x'A_z=(C, A, x, z), \ldots$$

The forms in question are

$$\begin{array}{l}
\{(B, A, x, y), \quad \ldots \quad (15')\\
(C, A, x, z),\\
(D, A, x, t),\\
(C, A, x, t)+2(N, A, x, z),\\
(D, A, x, z)+2(N, A, x, t),\\
(B, A, x, t)+2(M, A, x, y),\\
(D, A, x, y)+2(M, A, x, t),\\
(B, A, x, z)+2(F, A, x, y),\\
(C, A, x, y)+2(F, A, x, z),\\
(N, A, x, y)+(M, A, x, z)+(F, A, x, t)\}\!\!\!\int\!(y, z, t)^3
\end{array}$$

$$
\begin{array}{l}
\{(\mathrm{C},\ \mathrm{B},\ y,\ z),\\
\ (\mathrm{A},\ \mathrm{B},\ y,\ x),\\
\ (\mathrm{D},\ \mathrm{B},\ y,\ t),\\
\ (\mathrm{A},\ \mathrm{B},\ y,\ t)+2(\mathrm{L},\ \mathrm{B},\ y,\ x),\\
\ (\mathrm{D},\ \mathrm{B},\ y,\ x)+2(\mathrm{L},\ \mathrm{B},\ y,\ t),\\
\ (\mathrm{C},\ \mathrm{B},\ y,\ t)+2(\mathrm{M},\ \mathrm{B},\ y,\ z),\\
\ (\mathrm{D},\ \mathrm{B},\ y,\ z)+2(\mathrm{M},\ \mathrm{B},\ y,\ t),\\
\ (\mathrm{C},\ \mathrm{B},\ y,\ x)+2(\mathrm{G},\ \mathrm{B},\ y,\ z),\\
\ (\mathrm{A},\ \mathrm{B},\ y,\ z)+2(\mathrm{G},\ \mathrm{B},\ y,\ x),\\
\ (\mathrm{L},\ \mathrm{B},\ y,\ z)+\ (\mathrm{N},\ \mathrm{B},\ y,\ x)+(\mathrm{G},\ \mathrm{B},\ y,\ t)\}\,)\!(\,z,\ x,\ t)^3
\end{array}
$$

$$
\begin{array}{l}
\{(\mathrm{A},\ \mathrm{C},\ z,\ x),\\
\ (\mathrm{B},\ \mathrm{C},\ z,\ y),\\
\ (\mathrm{D},\ \mathrm{C},\ z,\ t),\\
\ (\mathrm{B},\ \mathrm{C},\ z,\ t)+2(\mathrm{M},\ \mathrm{C},\ z,\ y),\\
\ (\mathrm{D},\ \mathrm{C},\ z,\ y)+2(\mathrm{M},\ \mathrm{C},\ z,\ t),\\
\ (\mathrm{A},\ \mathrm{C},\ z,\ t)+2(\mathrm{L},\ \mathrm{C},\ z,\ x),\\
\ (\mathrm{D},\ \mathrm{C},\ z,\ x)+2(\mathrm{L},\ \mathrm{C},\ z,\ t),\\
\ (\mathrm{A},\ \mathrm{C},\ z,\ y)+2(\mathrm{H},\ \mathrm{C},\ z,\ x),\\
\ (\mathrm{B},\ \mathrm{C},\ z,\ x)+2(\mathrm{H},\ \mathrm{C},\ z,\ y),\\
\ (\mathrm{M},\ \mathrm{C},\ z,\ x)+\ (\mathrm{L},\ \mathrm{C},\ z,\ y)+(\mathrm{H},\ \mathrm{C},\ z,\ t)\}\,)\!(\,x,\ y,\ t)^3
\end{array}
$$

$$
\begin{array}{l}
\{(\mathrm{A},\ \mathrm{D},\ t,\ x),\\
\ (\mathrm{B},\ \mathrm{D},\ t,\ y),\\
\ (\mathrm{C},\ \mathrm{D},\ t,\ z),\\
\ (\mathrm{B},\ \mathrm{D},\ t,\ z)+2(\mathrm{F},\ \mathrm{D},\ t,\ y),\\
\ (\mathrm{C},\ \mathrm{D},\ t,\ y)+2(\mathrm{F},\ \mathrm{D},\ t,\ z),\\
\ (\mathrm{C},\ \mathrm{D},\ t,\ x)+2(\mathrm{G},\ \mathrm{D},\ t,\ z),\\
\ (\mathrm{A},\ \mathrm{D},\ t,\ z)+2(\mathrm{G},\ \mathrm{D},\ t,\ x),\\
\ (\mathrm{A},\ \mathrm{D},\ t,\ y)+2(\mathrm{H},\ \mathrm{D},\ t,\ x),\\
\ (\mathrm{B},\ \mathrm{D},\ t,\ x)+2(\mathrm{H},\ \mathrm{D},\ t,\ y),\\
\ (\mathrm{F},\ \mathrm{D},\ t,\ x)+\ (\mathrm{G},\ \mathrm{D},\ t,\ y)+(\mathrm{H},\ \mathrm{D},\ t,\ z)\}\,)\!(\,x,\ y,\ z)^3
\end{array}
$$

§ 2. *The identical relations between the forty-eight coordinates.*

On inspecting the expressions (13) and (14) it will be seen at once that the following relations subsist, viz.:

$$\begin{array}{ll} \cdot \quad N_y+M_z+F_t=0, & \cdot \quad N_y'+M_z'+F_t'=0, \\ N_x+\ \cdot\ +L_z+G_t=0, & N_x'+\ \cdot\ +L_z'+G_t'=0, \\ M_x+L_y+\ \cdot\ +H_t=0, & M_x'+L_y'+\ \cdot\ +H_t'=0, \\ F_x+G_y+H_z+\ \cdot\ =0\ ; & F_x'+G_y'+H_z'+\ \cdot\ =0. \end{array} \quad . \quad . \quad . \quad . \quad (16)$$

This is a set of $4+4=8$ identical relations between the forty-eight coordinates.

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The next set of identical relations is to be sought among the first minors of the discriminants of U, V, . . . , bordered respectively with the coefficients of P, Q, . . . Thus, selecting T, we have to examine the coefficients of A, B, . . . (say the quantities $\mathfrak{A}, \mathfrak{B}, \ldots$) in the development of

$$\begin{array}{llll} A, & H, & G, & D_x \\ H, & B, & F, & D_y \\ G, & F, & C, & D_z \\ D_x, & D_y, & D_z & \end{array} \quad (A, B, \ldots, \text{suffix } t).$$

We then have, changing the sign,

$$\begin{aligned} \mathfrak{A}&=BD_z^2-2FD_zD_y+CD_y^2 \\ &=\ |\beta, \beta', \delta| \times |\delta, \delta', \gamma|^2 \\ &\quad -\{|\beta, \gamma', \delta| \times |\gamma, \beta', \delta|\}\,|\delta, \delta', \gamma| \times |\delta, \delta', \beta| \\ &\quad +\ |\gamma, \gamma', \delta| \times |\delta, \delta', \beta|^2 \\ &=\{|\beta, \beta', \delta| \times |\delta, \delta', \gamma| - |\gamma, \beta', \delta| \times |\delta, \delta', \beta|\}\,|\delta, \delta', \gamma| \\ &\quad -\{|\beta, \gamma', \delta| \times |\delta, \delta', \gamma| - |\gamma, \gamma', \delta| \times |\delta, \delta', \beta|\}\,|\delta, \delta', \beta| \\ &=\begin{vmatrix} \beta, & \beta', & \delta, & \cdot & \cdot & \gamma \\ \beta_1, & \beta_1', & \delta_1, & \cdot & \cdot & \gamma_1 \\ \beta_2, & \beta_2', & \delta_2, & \cdot & \cdot & \gamma_2 \\ \beta, & \cdot & \cdot & \delta, & \delta', & \gamma \\ \beta_1, & \cdot & \cdot & \delta_1, & \delta_1', & \gamma_1 \\ \beta_2, & \cdot & \cdot & \delta_2, & \delta_2', & \gamma_2 \end{vmatrix} \times |\delta, \delta', \gamma| \times \begin{vmatrix} \gamma, & \gamma', & \delta & \cdot & \cdot & \beta \\ \gamma_1, & \gamma_1', & \delta_1, & \cdot & \cdot & \beta_1 \\ \gamma_2, & \gamma_2', & \delta_2, & \cdot & \cdot & \beta_2 \\ \gamma, & \cdot & \cdot & \delta, & \delta', & \beta \\ \gamma_1, & \cdot & \cdot & \delta_1, & \delta_1', & \beta_1 \\ \gamma_2, & \cdot & \cdot & \delta_2, & \delta_2', & \beta_2 \end{vmatrix} \end{aligned}$$

and if in these last determinants we subtract rows 4, 5, 6 from rows 1, 2, 3 respectively; and then subtract column 3 from column 4, we shall find that the whole expression

$$= |\beta, \gamma, \delta| \{ |\beta', \delta, \delta'| \times |\gamma, \delta, \delta'| - |\gamma', \delta, \delta'| \times |\beta, \delta, \delta'| \}$$
$$= - |\beta, \gamma, \delta| (\mathrm{D}, \mathrm{D}, y, z),$$

or more simply

$$= |\beta, \gamma, \delta| (\mathrm{D}, y, z).$$

From this result we may at once conclude the following group:

$$\mathfrak{A} = |\beta, \gamma, \delta| (\mathrm{D}, y, z) \quad \ldots \ldots \quad (17)$$
$$\mathfrak{B} = |\gamma, \alpha, \delta| (\mathrm{D}, z, x),$$
$$\mathfrak{C} = |\alpha, \beta, \delta| (\mathrm{D}, x, y).$$

Again, proceeding as before we should find

$$-2\mathfrak{F} = 2(\mathrm{AD}_y\mathrm{D}_z - \mathrm{HD}_z\mathrm{D}_x - \mathrm{GD}_x\mathrm{D}_y + \mathrm{FD}_x{}^2) \quad (\mathrm{A, H}, \ldots \text{ suffix } t)$$
$$= \{ |\alpha, \alpha', \delta| + |\alpha, \alpha', \delta| \} |\delta, \delta', \beta| \times |\delta, \delta', \gamma|$$
$$- \{ |\alpha, \beta', \delta| + |\beta, \alpha', \delta| \} |\delta, \delta', \alpha| \times |\delta, \delta', \gamma|$$
$$- \{ |\gamma, \alpha', \delta| + |\alpha, \gamma', \delta| \} |\delta, \delta', \beta| \times |\delta, \delta', \alpha|$$
$$+ \{ |\beta, \gamma', \delta| + |\gamma, \beta', \delta| \} |\delta, \delta', \alpha| \times |\delta, \delta', \alpha|$$

$$= \begin{vmatrix} \alpha, & \alpha', & \delta, & \cdot & \cdot & \beta \\ \alpha_1, & \alpha_1', & \delta_1, & \cdot & \cdot & \beta_1 \\ \alpha_2, & \alpha_2', & \delta_2, & \cdot & \cdot & \beta_2 \\ \alpha, & \cdot & \cdot & \delta, & \delta', & \beta \\ \alpha_1, & \cdot & \cdot & \delta_1, & \delta_1', & \beta_1 \\ \alpha_2, & \cdot & \cdot & \delta_2, & \delta_2', & \beta_2 \end{vmatrix} \times |\delta, \delta', \gamma| \times \begin{vmatrix} \alpha, & \alpha', & \delta, & \cdot & \cdot & \gamma \\ \alpha_1, & \alpha_1', & \delta_1, & \cdot & \cdot & \gamma_1 \\ \alpha_2, & \alpha_2', & \delta_2, & \cdot & \cdot & \gamma_2 \\ \alpha, & \cdot & \cdot & \delta, & \delta', & \gamma \\ \alpha_1, & \cdot & \cdot & \delta_1, & \delta_1', & \gamma_1 \\ \alpha_2, & \cdot & \cdot & \delta_2, & \delta_2', & \gamma_2 \end{vmatrix} \times |\delta, \delta', \beta|$$

$$- \left\{ \begin{vmatrix} \alpha, & \gamma', & \delta, & \cdot & \cdot & \beta \\ \alpha_1, & \gamma_1', & \delta_1, & \cdot & \cdot & \beta_1 \\ \alpha_2, & \gamma_2', & \delta_2, & \cdot & \cdot & \beta_2 \\ \alpha, & \cdot & \cdot & \delta, & \delta', & \beta \\ \alpha_1, & \cdot & \cdot & \delta_1, & \delta_1', & \beta_1 \\ \alpha_2, & \cdot & \cdot & \delta_2, & \delta_2', & \beta_2 \end{vmatrix} + \begin{vmatrix} \alpha, & \beta', & \delta, & \cdot & \cdot & \gamma \\ \alpha_1, & \beta_1', & \delta_1, & \cdot & \cdot & \gamma_1 \\ \alpha_2, & \beta_2', & \delta_2', & \cdot & \cdot & \gamma_2 \\ \alpha, & \cdot & \cdot & \delta, & \delta', & \gamma \\ \alpha_1, & \cdot & \cdot & \delta_1, & \delta_1', & \gamma_1 \\ \alpha_2, & \cdot & \cdot & \delta_2, & \delta_2', & \gamma_2 \end{vmatrix} \right\} \times |\delta, \delta', \alpha|$$

$$= - |\delta, \delta', \alpha'| \times |\delta, \delta', \gamma| \times |\alpha, \delta, \beta| - |\delta, \delta', \alpha'| \times |\delta, \delta', \beta| \times |\alpha, \delta, \gamma|$$
$$+ |\delta, \delta', \alpha| \times |\delta, \delta', \gamma'| \times |\alpha, \delta, \beta| + |\delta, \delta', \alpha| \times |\delta, \delta', \beta'| \times |\alpha, \delta, \gamma|$$

$$= |\alpha, \delta, \beta| (\mathrm{D}, z, x) + |\gamma, \alpha, \delta| (\mathrm{D}, x, y).$$

From this we may conclude the following group,

$$
\begin{aligned}
2\mathfrak{F} &= |\alpha, \beta, \delta|\,(\mathrm{D}, z, x) + |\gamma, \alpha, \delta|\,(\mathrm{D}, x, y) \quad \ldots \quad (18)\\
2\mathfrak{G} &= |\alpha, \beta, \delta|\,(\mathrm{D}, y, z) + |\beta, \gamma, \delta|\,(\mathrm{D}, x, y),\\
2\mathfrak{H} &= |\gamma, \alpha, \delta|\,(\mathrm{D}, y, z) + |\beta, \gamma, \delta|\,(\mathrm{D}, z, x).
\end{aligned}
$$

From (17) and (18) we can now eliminate the three quantities $|\beta, \gamma, \delta|$, $|\gamma, \alpha, \delta|$, $|\alpha, \beta, \delta|$, and obtain the following identical relations,

$$
\begin{aligned}
&\mathfrak{B}(\mathrm{D}, x, y)^2 - 2\mathfrak{F}(\mathrm{D}, x, y)(\mathrm{D}, z, x) + \mathfrak{C}(\mathrm{D}, z, x)^2 = 0 \quad \ldots \quad (19)\\
&\mathfrak{C}(\mathrm{D}, y, z)^2 - 2\mathfrak{G}(\mathrm{D}, y, z)(\mathrm{D}, x, y) + \mathfrak{A}(\mathrm{D}, x, y)^2 = 0\\
&\mathfrak{A}(\mathrm{D}, z, x)^2 - 2\mathfrak{H}(\mathrm{D}, z, x)(\mathrm{D}, y, z) + \mathfrak{B}(\mathrm{D}, y, z)^2 = 0
\end{aligned}
$$

to these may be added

$$
\begin{aligned}
&-\mathfrak{F}(\mathrm{D}, y, z)^2 - \mathfrak{A}(\mathrm{D}, z, x)(\mathrm{D}, x, y) + \mathfrak{H}(\mathrm{D}, x, y)(\mathrm{D}, y, z) + \mathfrak{G}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0 \quad . \quad (20)\\
&-\mathfrak{G}(\mathrm{D}, z, x)^2 + \mathfrak{H}(\mathrm{D}, z, x)(\mathrm{D}, x, y) - \mathfrak{B}(\mathrm{D}, x, y)(\mathrm{D}, y, z) + \mathfrak{F}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0\\
&-\mathfrak{H}(\mathrm{D}, x, y)^2 + \mathfrak{G}(\mathrm{D}, z, x)(\mathrm{D}, x, y) + \mathfrak{F}(\mathrm{D}, x, y)(\mathrm{D}, y, z) - \mathfrak{C}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0
\end{aligned}
$$

But, as the six equations (19) and (20) are in any case equivalent to only three independent conditions, it is not necessary to go beyond the equations (19).

[POSTSCRIPT.

Added April 23, 1881.

In the present case however the equations (19) are themselves not independent, as may be shown in the following way. Write for the moment

$$\mathrm{D}_x = \mathrm{X},\ \mathrm{D}_y = \mathrm{Y},\ \mathrm{D}_z = \mathrm{Z};\ \mathrm{D}_x' = \mathrm{X}',\ \mathrm{D}_y' = \mathrm{Y}',\ \mathrm{D}_z' = \mathrm{Z}';$$

then

$$(\mathrm{D}, y, z) = \mathrm{Y}'\mathrm{Z} - \mathrm{Y}\mathrm{Z}',\ (\mathrm{D}, z, x) = \mathrm{Z}'\mathrm{X} - \mathrm{Z}\mathrm{X}',\ (\mathrm{D}, x, y) = \mathrm{X}'\mathrm{Y} - \mathrm{X}\mathrm{Y}'.$$

Inserting these values in the first equations of (19) we obtain

$$
\begin{aligned}
&\mathfrak{B}(\mathrm{X}'\mathrm{Y} - \mathrm{X}\mathrm{Y}')^2 - 2\mathfrak{F}(\mathrm{X}'\mathrm{Y} - \mathrm{X}\mathrm{Y}')(\mathrm{Z}'\mathrm{X} - \mathrm{Z}\mathrm{X}') + \mathfrak{C}(\mathrm{Z}'\mathrm{X} - \mathrm{Z}\mathrm{X}')^2 = 0\\
&= (\mathfrak{B}\mathrm{Y}^2 + 2\mathfrak{F}\mathrm{YZ} + \mathfrak{C}\mathrm{Z}^2)\mathrm{X}'^2 + \mathfrak{B}\mathrm{X}^2.\mathrm{Y}'^2 + \mathfrak{C}\mathrm{X}^2.\mathrm{Z}'^2 + 2\mathfrak{F}\mathrm{X}^2.\mathrm{Y}'\mathrm{Z}'\\
&\quad - 2\mathfrak{C}\mathrm{ZX}.\mathrm{Z}'\mathrm{X}' - 2\mathfrak{B}\mathrm{XY}.\mathrm{X}'\mathrm{Y}' - 2\mathfrak{F}\mathrm{XY}.\mathrm{Z}'\mathrm{X}' - 2\mathfrak{F}\mathrm{ZX}.\mathrm{X}'\mathrm{Y}'.
\end{aligned}
$$

But

$$
\begin{aligned}
&\mathfrak{B}\mathrm{Y}^2 + 2\mathfrak{F}\mathrm{YZ} + \mathfrak{C}\mathrm{Z}^2\\
&= (\mathrm{CX}^2 - 2\mathrm{GZX} + \mathrm{AZ}^2)\mathrm{Y}^2 - 2(\mathrm{AYZ} - \mathrm{HZX} - \mathrm{GXY} + \mathrm{FX}^2)\mathrm{YZ} + (\mathrm{AY}^2 - 2\mathrm{HXY} + \mathrm{BX}^2)\mathrm{Z}^2\\
&= (\mathrm{CY}^2 - 2\mathrm{FYZ} + \mathrm{BZ}^2)\mathrm{X}^2\\
&= \mathfrak{A}\mathrm{X}^2
\end{aligned}
$$

also

$$\begin{aligned}&\mathfrak{C}Z+\mathfrak{F}Y\\ &=(AY^2-2HXY+BX^2)Z-(AYZ-HZX-GXY+FX^2)Y\\ &=(BZX-FXY-HYZ+GY^2)X\\ &=-\mathfrak{G}X\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{B}Y+\mathfrak{F}Z\\ &=(CX^2-2GZX+AZ^2)Y-(AYZ-HZX-GXY+FX^2)Z\\ &=(CXY-GYZ-FZX+HZ^2)X\\ &=-\mathfrak{H}X\end{aligned}$$

Hence the whole expression

$$=(\mathfrak{A}X'^2+\mathfrak{B}Y'^2+\mathfrak{C}Z'^2+2\mathfrak{F}Y'Z'+2\mathfrak{G}Z'X'+2\mathfrak{H}X'Y')X^2,$$

or

$$(\mathfrak{A}, \mathfrak{B}, \ldots)(X', Y', Z')^2=0$$

i.e.,

$$\begin{vmatrix} A, & H, & G, & D_x, & D_x' \\ H, & B, & F, & D_y, & D_y' \\ G, & F, & C, & D_z, & D_z' \\ D_x, & D_y, & D_z, & . & . \\ D_x', & D_y', & D_z', & . & . \end{vmatrix}=0 \quad . \quad . \quad . \quad (21)$$

The equations (19), and consequently also the equations (19) and (20), are therefore together equivalent only to the single condition (21).

Now the relation (21) has been derived from the form T; if to this we add the corresponding relations derived from the forms U, V, W, we shall have four relations. Others are readily obtained as follows. If we form the four systems of which (17) is one, we may write them down thus:—

	$\mathfrak{B}=\lvert\gamma,\alpha,\delta\rvert(A, z, t)$,	$\mathfrak{C}=\lvert\alpha,\beta,\delta\rvert(A, y, t)$,	$\mathfrak{D}=\lvert\alpha,\beta,\gamma\rvert(A, y, z)$,	(suffix x)
$\mathfrak{A}=\lvert\beta,\gamma,\delta\rvert(B, z, t)$,	.	$\mathfrak{C}=\lvert\alpha,\beta,\delta\rvert(B, z, t)$,	$\mathfrak{D}=\lvert\alpha,\beta,\gamma\rvert(B, z, x)$,	(„ y)
$\mathfrak{A}=\lvert\beta,\gamma,\delta\rvert(C, y, t)$,	$\mathfrak{B}=\lvert\gamma,\alpha,\delta\rvert(C, x, t)$,	.	$\mathfrak{D}=\lvert\alpha,\beta,\gamma\rvert(C, x, y)$,	(„ z)
$\mathfrak{A}=\lvert\beta,\gamma,\delta\rvert(D, y, z)$,	$\mathfrak{B}=\lvert\gamma,\alpha,\delta\rvert(D, z, x)$,	$\mathfrak{C}=\lvert\alpha,\beta,\delta\rvert(D, x, y)$,		(„ t)

Whence

$$\begin{array}{cccc} . & \mathfrak{A}:(B,z,t)= & \mathfrak{A}:(C,y,t)= & \mathfrak{A}:(D,y,z), \\ \mathfrak{B}:(A,z,t)= & . & =\mathfrak{B}:(C,x,t)= & \mathfrak{B}:(D,z,x), \\ \mathfrak{C}:(A,y,t)= & \mathfrak{C}:(B,x,t)= & . & =\mathfrak{C}:(D,x,y), \\ \mathfrak{D}:(A,y,z)= & \mathfrak{D}:(B,z,x)= & \mathfrak{D}:(C,x,y)= & . \\ \text{(suffix } x) & \text{(suffix } y) & \text{(suffix } z) & \text{(suffix } t) \end{array} \quad . \quad . \quad (22)$$

We thus have eight more relations. And if to these we add the corresponding relations derived from the forms U′, V′, W′, T′, we shall have 2(4+8)=24 relations in addition to (16), viz., 8+24=32 in all.]

To obtain two more, write for the moment

$$|\beta, \gamma, \delta| = p^{-1}, \quad |\gamma, \alpha, \delta| = q^{-1}, \quad |\alpha, \beta, \delta| = r^{-1}, \quad |\alpha, \beta, \gamma| = s^{-1} \quad . \quad (23)$$

and then multiply (17) by pD_x, qD_y, rD_z respectively and add them together. This will give

$$\mathfrak{A}pD_x + \mathfrak{B}qD_y + \mathfrak{C}rD_z = 0.$$

And if we proceed in a similar way with the corresponding systems derived from U, V, W respectively, we may form the following system

$$\begin{array}{lll} \cdot \quad \mathfrak{B}A_y q + \mathfrak{C}A_z r - \mathfrak{D}A_t s = 0, & \mathfrak{A}, \mathfrak{B}, \ldots \text{ suffix } x & \quad . \; . \; . \quad (24) \\ \mathfrak{A}B_x p + \quad \cdot \quad + \mathfrak{C}B_z r - \mathfrak{D}B_t s = 0, & \text{,,} \qquad \text{,,} \quad y, & \\ \mathfrak{A}C_x p + \mathfrak{B}C_y q + \quad \cdot \quad - \mathfrak{D}C_t s = 0, & \text{,,} \qquad \text{,,} \quad z, & \\ \mathfrak{A}D_x p + \mathfrak{B}D_y q + \mathfrak{C}D_z r + \quad \cdot \quad = 0, & \text{,,} \qquad \text{,,} \quad t, & \end{array}$$

whence eliminating p, q, r, s, we finally obtain the following relation,

$$\begin{array}{llll} \cdot \quad \mathfrak{B}A_y, & \mathfrak{C}A_z, & \mathfrak{D}A_t = 0, & \mathfrak{A}, \mathfrak{B}, \ldots \text{ suffix } x \; . \; . \; . \; . \; . \quad (25) \\ \mathfrak{A}B_x, \quad \cdot & \mathfrak{C}B_z, & \mathfrak{D}B_t & \text{,,} \qquad \text{,,} \quad y \\ \mathfrak{A}C_x, \; \mathfrak{B}C_y, & \quad \cdot & \mathfrak{D}C_t & \text{,,} \qquad \text{,,} \quad z \\ \mathfrak{A}D_x, \; \mathfrak{B}D_y, & \mathfrak{C}D_z, & \quad \cdot & \text{,,} \qquad \text{,,} \quad t \end{array}$$

To this may be added the corresponding relation obtained from the forms U′, V′, W′, T′. These added to the former conditions give us $32+2=34$.

It was however remarked at the outset that the equations $UP'-U'P=0$, &c., are lineo-linear in the U coordinates, and also in the U′ coordinates; and as we are concerned with the ratios only of the coefficients, and not with their absolute values, we are in fact concerned only with the ratios of the U coordinates *inter se*, and with those of the U′ coordinates *inter se*, and not with their absolute values. Hence the number of independent coordinates will finally be reduced to

$$48-34-2=12,$$

as it should be.